Determination of the reference symmetry axis of a generally anisotropic medium which is approximately transversely isotropic

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ABSTRACT

For a given stiffness tensor (tensor of elastic moduli) of a generally anisotropic medium, we estimate to what extent the medium is transversely isotropic (uniaxial) and determine the direction of its reference symmetry axis expressed in terms of the unit reference symmetry vector. If the medium is exactly transversely isotropic (exactly uniaxial), we obtain the direction of its symmetry axis. We can also calculate the first–order and second–order spatial derivatives of the reference symmetry vector which may be useful in tracing the reference rays for the coupling ray theory. The proposed method is tested using various transversely isotropic (uniaxial) and approximately transversely isotropic (approximately uniaxial) media.

Keywords: elastic anisotropy, stiffness tensor, elastic moduli, transverse isotropy, approximate transverse isotropy, reference symmetry axis

1. INTRODUCTION

The coupling ray theory (*Coates and Chapman, 1990*; *Bulant and Klimeš, 2002*; *Klimeš and Bulant, 2012*) is usually applied to common anisotropic rays (*Bakker, 2002*; *Klimeš and Bulant, 2004, 2006*; *Klimeš, 2006*; *Bulant and Klimeš, 2008*). However, the coupling ray theory is more accurate if it is applied to reference rays which are closer to the actual S–wave paths (*Klimeš and Bulant, 2014a, 2015*).

If we know that a given medium is transversely isotropic (uniaxial), we can separate the slowness surface into the P–wave slowness sheet, the SH–wave slowness sheet and the SV–wave slowness sheet. We may then trace the SH rays and SV rays (*Klimeš and Bulant, 2014a*), and use them as the reference rays for the prevailing–frequency approximation of the coupling ray theory. In this case, the SH rays and SV rays are better reference rays than the common anisotropic reference rays.

Even if a given medium is not transversely isotropic but is approximately transversely isotropic (approximately uniaxial), the SH and SV reference rays (*Klimeš and Bulant, 2015*) may represent better reference rays than the common anisotropic reference rays. Note that, in this case, the anisotropic–ray–theory rays often cannot be used as the reference rays (*Bulant and Klimeš, 2014*; *Klimeš and Bulant, 2014b*).

For a given stiffness tensor (tensor of elastic moduli) of a generally anisotropic medium, it is thus very useful to be able to estimate to what extent the medium is transversely isotropic and to determine the direction of its reference symmetry axis. This task is not only important for the accuracy of the coupling ray theory as explained above, but simultaneously represents a general problem of the *nearest media approximation*.

The stiffness tensor of a transversely isotropic medium is independent of the rotation around the symmetry axis.

For a given stiffness tensor of a generally anisotropic medium and a given rotation axis, we calculate the derivative of the stiffness tensor with respect to the angle of rotation in Section 2.1. The Frobenius norm of the derivative of the stiffness tensor with respect to the angle of rotation divided by the Frobenius norm of the stiffness tensor characterizes the strength of the dependence of the stiffness tensor on the rotation.

In Section 2.2, we determine the rotation axis by minimizing the Frobenius norm of the angular derivative of the stiffness tensor and refer to it as the reference symmetry axis. The direction of the reference symmetry axis is specified in terms of the reference symmetry vector. The stiffness tensor is axially rotated about a candidate reference symmetry axis, and the squared Frobenius norm of the derivative with respect to the rotation angle is expressed in terms of a quadratic form. The reference symmetry vector then appears as the eigenvector of this quadratic form associated with the minimum eigenvalue. We thus offer an analytically differentiable analytical solution of the problem which was initially solved by brute–force numerical methods (*Norris, 2006*; *Moakher and Norris, 2006*; *Kochetov and Slawinski, 2008*) or by very rough approximations (*Arts et al., 1991*).

In Sections 2.3 and 2.4, we also calculate the first–order and second–order spatial derivatives of the reference symmetry vector, which may be useful in tracing the SH and SV reference rays (*Klimeš and Bulant, 2015*) in heterogeneous velocity models with spatially varying reference symmetry vector, and for solving the corresponding equations of geodesic deviation (dynamic ray tracing equations).

The proposed method is tested in various transversely isotropic and approximately transversely isotropic media in Section 3.

The lower–case Roman indices take values 1, 2 and 3. The upper–case Roman indices take values 1 and 2. Indices in parentheses are used to index the eigenvalues and corresponding eigenvectors. The Einstein summation over repetitive lower–case Roman indices (without parentheses) corresponding to the 3 spatial coordinates, is used throughout the paper.

2. REFERENCE SYMMETRY AXIS

2.1. Derivative of the stiffness tensor with respect to the angle of rotation

Transformation matrix $R_{in}(\varphi, t_a)$ corresponding to the rotation of vectors about a given unit vector t_a by angle φ is an orthogonal matrix, with $R_{in}(0, t_a) = \delta_{in}$, where Kronecker delta δ_{in} represents the elements of the identity matrix. The derivative $\omega_{in} = R'_{in} = \mathrm{d}R_{in}/\mathrm{d}\varphi$ of the transformation matrix at $\varphi = 0$ reads

$$\omega_{in}(0, t_a) = -t_m \varepsilon_{min} \ , \tag{1}$$

where ε_{ijk} is the Levi–Civita symbol.

The rotated stiffness tensor reads

$$a_{ijkl}(\varphi, t_a) = R_{ip}(\varphi, t_a)\, R_{jq}(\varphi, t_b)\, R_{kr}(\varphi, t_c)\, R_{ls}(\varphi, t_d)\, a_{pqrs} \ , \tag{2}$$

where a_{pqrs} without arguments is the non–rotated tensor. Since rotation vector t_a is unit, the derivative $a'_{ijkl} = \mathrm{d}a_{ijkl}/\mathrm{d}\varphi(0, t_a)$ of stiffness tensor $a_{ijkl}(\varphi, t_a)$ with respect to the angle φ of rotation at $\varphi = 0$ follows directly from transformation (2),

$$a'_{ijkl} = \omega_{in} a_{njkl} + \omega_{jn} a_{inkl} + \omega_{kn} a_{ijnl} + \omega_{ln} a_{ijkn} \ . \tag{3}$$

We insert matrix (1) into angular derivative (3) and obtain

$$a'_{ijkl} = -t_m \left(\varepsilon_{min} a_{njkl} + \varepsilon_{mjn} a_{inkl} + \varepsilon_{mkn} a_{ijnl} + \varepsilon_{mln} a_{ijkn} \right) . \tag{4}$$

We define tensor

$$d_{ijklm} = \varepsilon_{min} a_{njkl} + \varepsilon_{mjn} a_{inkl} + \varepsilon_{mkn} a_{ijnl} + \varepsilon_{mln} a_{ijkn} \tag{5}$$

and express angular derivative (4) as

$$a'_{ijkl} = -d_{ijklm}\, t_m \ . \tag{6}$$

Note that if we put $a'_{ijkl} = 0$ here, we obtain the system of equations for the stiffness tensor of a transversely isotropic medium, equivalent to the equations of *Cowin and Mehrabadi (1987)*.

2.2. Reference symmetry vector

We choose the square

$$y = a'_{ijkl}\, a'_{ijkl} \tag{7}$$

of the Frobenius norm of the derivative of the stiffness tensor with respect to the angle of rotation as the objective function. We insert derivative (6) into objective function (7) and obtain

$$y = t_m B_{mn} t_n \ , \tag{8}$$

where

$$B_{mn} = d_{ijklm}\, d_{ijkln} \ . \tag{9}$$

The minimum value of objective function (8) over all unit vectors t_m is attained for the eigenvector $t_{i(3)}$ of matrix B_{mn} corresponding to the smallest eigenvalue $B_{(3)}$.

We shall refer to this eigenvector $t_{i(3)}$ as the *reference symmetry vector* and to the corresponding direction as the *reference symmetry axis*.

The ratio

$$\rho = \sqrt{\frac{a'_{ijkl}\, a'_{ijkl}}{a_{ijkl}\, a_{ijkl}}} \tag{10}$$

of the Frobenius norm $\sqrt{a'_{ijkl}\, a'_{ijkl}}$ of the derivative of the stiffness tensor with respect to the angle of rotation and the Frobenius norm $\sqrt{a_{ijkl}\, a_{ijkl}}$ of the stiffness tensor characterizes how strongly the stiffness tensor depends on the rotation. For the reference symmetry vector $t_i = t_{i(3)}$, ratio (10) reads

$$\rho_{(3)} = \sqrt{\frac{B_{(3)}}{a_{ijkl}\, a_{ijkl}}} \ . \tag{11}$$

This ratio characterizes the extent to which the medium is not transversely isotropic. We shall thus refer to it as the *non–TI ratio*.

Note that the reference symmetry vector $t_{i(3)}$ is stable and has a good physical meaning only if the minimum eigenvalue $B_{(3)}$ of matrix B_{mn} is considerably smaller than the two other eigenvalues $B_{(1)}$ and $B_{(2)}$, i.e. if $\rho_{(3)}$ is considerably smaller than ratios

$$\rho_{(A)} = \sqrt{\frac{B_{(A)}}{a_{ijkl}\, a_{ijkl}}} \ . \tag{12}$$

If non–TI ratio (11) is zero, the medium is exactly transversely isotropic and reference symmetry vector $t_{i(3)}$ specifies its symmetry axis.

2.3. First–order spatial derivatives of the symmetry vector

The symmetry vector may vary with spatial coordinates. If we use the spatially varying symmetry vector for tracing the SH and SV reference rays (*Klimeš and Bulant, 2015*), we need its first–order spatial derivatives.

Since the symmetry vector is a unit eigenvector of matrix (9), we can calculate its first–order and second–order partial derivatives with respect to spatial coordinates analogously to the derivatives of the eigenvectors of the Christoffel matrix.

In addition to reference symmetry vector $t_{i(3)}$ corresponding to the minimum eigenvalue $B_{(3)}$ of matrix (9), we introduce also the two other unit eigenvectors $t_{i(A)}$ corresponding to eigenvalues $B_{(A)}$.

The first–order partial derivatives of tensor (5) with respect to spatial coordinates read

$$d_{ijklm,p} = \varepsilon_{min} a_{njkl,p} + \varepsilon_{mjn} a_{inkl,p} + \varepsilon_{mkn} a_{ijnl,p} + \varepsilon_{mln} a_{ijkn,p} \ . \tag{13}$$

The first–order partial derivatives of matrix (9) with respect to spatial coordinates then read

$$B_{mn,p} = d_{ijklm,p}\, d_{ijkln} + d_{ijklm}\, d_{ijkln,p} \ . \tag{14}$$

We transform the first–order partial derivatives of matrix (9) into eigenvectors $t_{i(a)}$,

$$B_{(ab),p} = t_{m(a)} B_{mn,p} t_{n(b)} \ . \tag{15}$$

Hereinafter, the subscripts in parentheses denote the covariant transform (15) into eigenvectors $t_{i(a)}$. The first–order partial derivatives of eigenvector $t_i = t_{i(3)}$ with respect to spatial coordinates then read (*Klimeš, 2006, Eq. 17*)

$$t_{i,p} = \sum_A t_{i(A)} \frac{B_{(A3),p}}{B_{(3)} - B_{(A)}} \ . \tag{16}$$

2.4. Second–order spatial derivatives of the symmetry vector

If we use the spatially varying symmetry vector for tracing the SH and SV reference rays (*Klimeš and Bulant, 2015*), we need its second–order spatial derivatives for calculating the geodesic deviation (dynamic ray tracing).

The second–order partial derivatives of tensor (5) with respect to spatial coordinates read

$$d_{ijklm,pq} = \varepsilon_{min} a_{njkl,pq} + \varepsilon_{mjn} a_{inkl,pq} + \varepsilon_{mkn} a_{ijnl,pq} + \varepsilon_{mln} a_{ijkn,pq} \ . \tag{17}$$

The second–order partial derivatives of matrix (9) with respect to spatial coordinates then read

$$B_{mn,pq} = d_{ijklm,pq}\, d_{ijkln} + d_{ijklm}\, d_{ijkln,pq} + d_{ijklm,p}\, d_{ijkln,q} + d_{ijklm,q}\, d_{ijkln,p} \ . \tag{18}$$

We transform the second–order partial derivatives of matrix (9) into eigenvectors $t_{i(A)}$,

$$B_{(ab),pq} = t_{m(a)}\, B_{mn,pq}\, t_{n(b)} \ . \tag{19}$$

The second–order partial derivatives of eigenvector $t_i = t_{i(3)}$ with respect to spatial coordinates then read (*Klimeš and Bulant, 2015, Eqs 39–40*)

$$\begin{aligned} t_{i,pq} = \sum_A t_{i(A)} \Bigg(& \frac{B_{(A3),pq}}{B_{(3)} - B_{(A)}} - \frac{B_{(A3),p} B_{(33),q} + B_{(A3),q} B_{(33),p}}{(B_{(3)} - B_{(A)})^2} \\ & + \sum_B \frac{B_{(AB),p} B_{(B3),q} + B_{(AB),q} B_{(B3),p}}{(B_{(3)} - B_{(A)})\,(B_{(3)} - B_{(B)})} \Bigg) \\ & - t_{i(3)} \sum_B \frac{B_{(B3),p} B_{(B3),q}}{(B_{(3)} - B_{(B)})^2} \ . \end{aligned} \tag{20}$$

3. NUMERICAL EXAMPLES

Unit reference symmetry vector $t_{i(3)}$ and non–TI ratio (11) for a given stiffness tensor a_{ijkl} is determined according to Sections 2.1 and 2.2 of this paper by new program `tiaxis` (*Bucha and Bulant, 2015*).

3.1. Velocity model WA

The density reduced stiffness tensor in the vertically heterogeneous 1–D anisotropic velocity model WA by *Pšenčík and Dellinger (2001)* at the surface (zero depth) reads

$$\begin{array}{c} \\ 11 \\ 22 \\ 33 \\ 23 \\ 13 \\ 12 \end{array} \begin{array}{c} \begin{array}{cccccc} \quad 11 & \quad 22 & \quad 33 & \quad 23 & \quad 13 & \quad 12 \end{array} \\ \begin{pmatrix} 13.39 & 4.46 & 4.46 & 0.00 & 0.00 & 0.00 \\ & 15.71 & 5.04 & 0.00 & 0.00 & 0.00 \\ & & 15.71 & 0.00 & 0.00 & 0.00 \\ & & & 5.33 & 0.00 & 0.00 \\ & & & & 4.98 & 0.00 \\ & & & & & 4.98 \end{pmatrix} \end{array} . \tag{21}$$

Non–TI ratio (11) determined for this medium is

$$\rho_{(3)} = 0.000847 , \tag{22}$$

and the corresponding unit reference symmetry vector is

$$t_{i(3)} = (1.000000 \ \ 0.000000 \ \ 0.000000) . \tag{23}$$

We see that the medium is not exactly transversely isotropic but is approximately transversely isotropic.

We now change density reduced stiffness tensor (21) slightly to density reduced stiffness tensor

$$\begin{array}{c} \\ 11 \\ 22 \\ 33 \\ 23 \\ 13 \\ 12 \end{array} \begin{array}{c} \begin{array}{cccccc} \quad 11 & \quad 22 & \quad 33 & \quad 23 & \quad 13 & \quad 12 \end{array} \\ \begin{pmatrix} 13.39 & 4.46 & 4.46 & 0.00 & 0.00 & 0.00 \\ & 15.70 & 5.04 & 0.00 & 0.00 & 0.00 \\ & & 15.70 & 0.00 & 0.00 & 0.00 \\ & & & 5.33 & 0.00 & 0.00 \\ & & & & 4.98 & 0.00 \\ & & & & & 4.98 \end{pmatrix} \end{array} \tag{24}$$

of a transversely isotropic medium. Non–TI ratio (11) determined for this medium is

$$\rho_{(3)} = 0.000000 , \tag{25}$$

and the corresponding unit reference symmetry vector is

$$t_{i(3)} = (1.000000 \ \ 0.000000 \ \ 0.000000) . \tag{26}$$

3.2. Velocity model QI

Velocity model WA was rotated by 45° about the positive x_3 half–axis in order to create vertically heterogeneous 1–D anisotropic velocity model QI. The density reduced stiffness tensor in $\mathrm{km}^2\mathrm{s}^{-2}$ in velocity model QI at the surface (zero depth) reads (*Bulant and Klimeš, 2002, Eq. 38*; *Klimeš and Bulant, 2004, Eq. 57*; *Pšenčík et al., 2012, Eq. 16*)

$$\begin{array}{c} \\ 11 \\ 22 \\ 33 \\ 23 \\ 13 \\ 12 \end{array} \begin{array}{c} \begin{array}{cccccc} 11 & 22 & 33 & 23 & 13 & 12 \end{array} \\ \begin{pmatrix} 14.485 & 4.525 & 4.750 & 0.000 & 0.000 & -0.580 \\ & 14.485 & 4.750 & 0.000 & 0.000 & -0.580 \\ & & 15.710 & 0.000 & 0.000 & -0.290 \\ & & & 5.155 & -0.175 & 0.000 \\ & & & & 5.155 & 0.000 \\ & & & & & 5.045 \end{pmatrix} \end{array} . \tag{27}$$

Non–TI ratio (11) determined for this medium is

$$\rho_{(3)} = 0.000847 \ , \tag{28}$$

and the corresponding unit reference symmetry vector is

$$t_{i(3)} = (0.707107 \;\; 0.707107 \;\; 0.000000) \ . \tag{29}$$

We see that the medium is not exactly transversely isotropic but is approximately transversely isotropic, analogously to velocity model WA.

3.3. Velocity model KISS

Velocity model WA was rotated by 1° about the positive x_3 half–axis in order to create the vertically heterogeneous 1–D anisotropic velocity model KISS. The density reduced stiffness tensor in $\mathrm{km}^2\mathrm{s}^{-2}$ in velocity model KISS at the surface (zero depth) reads (*Pšenčík et al., 2012, Eq. 20*)

$$\begin{array}{c} \\ 11 \\ 22 \\ 33 \\ 23 \\ 13 \\ 12 \end{array} \begin{array}{c} \begin{array}{cccccc} 11 & 22 & 33 & 23 & 13 & 12 \end{array} \\ \begin{pmatrix} 13.39063 & 4.46008 & 4.46018 & 0.00000 & 0.00000 & -.01797 \\ & 15.70921 & 5.03982 & 0.00000 & 0.00000 & -.02251 \\ & & 15.71000 & 0.00000 & 0.00000 & -.01012 \\ & & & 5.32989 & -.00611 & 0.00000 \\ & & & & 4.98011 & 0.00000 \\ & & & & & 4.98008 \end{pmatrix} \end{array} . \tag{30}$$

Non–TI ratio (11) determined for this medium is

$$\rho_{(3)} = 0.000848 \ , \tag{31}$$

and the corresponding unit reference symmetry vector is

$$t_{i(3)} = (0.999848 \;\; 0.017452 \;\; 0.000000) \ . \tag{32}$$

We see that the medium is not exactly transversely isotropic but is approximately transversely isotropic, analogously to velocity model WA.

3.4. Velocity model SC1_II

The density reduced stiffness tensor in homogeneous anisotropic velocity model 1 by *Shearer and Chapman (1989)* reads

$$\begin{array}{c} \\ 11 \\ 22 \\ 33 \\ 23 \\ 13 \\ 12 \end{array} \begin{array}{c} \begin{array}{cccccc} 11 & 22 & 33 & 23 & 13 & 12 \end{array} \\ \begin{pmatrix} 20.04 & 7.41 & 7.41 & 0.00 & 0.00 & 0.00 \\ & 20.22 & 7.46 & 0.00 & 0.00 & 0.00 \\ & & 20.22 & 0.00 & 0.00 & 0.00 \\ & & & 6.38 & 0.00 & 0.00 \\ & & & & 5.10 & 0.00 \\ & & & & & 5.10 \end{pmatrix} \end{array} . \tag{33}$$

Non–TI ratio (11) determined for this medium is

$$\rho_{(3)} = 0.000000 \ , \tag{34}$$

and the corresponding unit reference symmetry vector is

$$t_{i(3)} = (1.000000 \ \ 0.000000 \ \ 0.000000) \ . \tag{35}$$

We see that the medium is transversely isotropic within the rounding errors. If we inspect stiffness tensor (33) manually, we see that the medium is exactly transversely isotropic.

Velocity model 1 by *Shearer and Chapman (1989)* was first rotated by 45° about the positive x_2 half–axis and then rotated by 30° about the positive x_3 half–axis in order to create the stiffness tensor of the vertically heterogeneous 1–D anisotropic velocity model SC1_II at the surface (zero depth). After these rotations, the symmetry vector should read

$$t_{i(3)} = \left(\sqrt{3/8} \ \ \sqrt{1/8} \ \ \sqrt{1/2} \right) \ . \tag{36}$$

The density reduced stiffness tensor in km^{2}s^{-2} in velocity model SC1_II at the surface (zero depth) reads (*Pšenčík et al., 2012, Eq. 19*)

$$\begin{array}{c} \\ 11 \\ 22 \\ 33 \\ 23 \\ 13 \\ 12 \end{array} \begin{array}{c} \begin{array}{cccccc} 11 & 22 & 33 & 23 & 13 & 12 \end{array} \\ \begin{pmatrix} 18.97125 & 7.67125 & 8.36125 & 0.46000 & -0.31177 & -0.15589 \\ & 19.64625 & 7.74375 & -0.49500 & 0.25115 & -0.42868 \\ & & 18.87000 & -0.02250 & -0.03897 & 0.53477 \\ & & & 5.89500 & 0.26847 & -0.28146 \\ & & & & 6.20500 & 0.15250 \\ & & & & & 5.97625 \end{pmatrix} \end{array} . \tag{37}$$

Non–TI ratio (11) determined for this medium is

$$\rho_{(3)} = 0.000054 \ , \tag{38}$$

and the corresponding unit reference symmetry vector is

$$t_{i(3)} = (0.612372 \ \ 0.353554 \ \ 0.707107) \ . \tag{39}$$

We see that the medium is not exactly transversely isotropic but is close to transversely isotropic at the surface (zero depth). The difference from an exactly transversely isotropic medium is caused by the rounding errors in stiffness tensor (37).

Numerically determined unit reference symmetry vector (39) is equal to its theoretical estimate (36).

At the depth of 1.5 km, velocity model SC1_II is very close to isotropic, but is slightly cubic and its symmetry axes coincide with the coordinate axes. The density reduced stiffness tensor in $\mathrm{km}^2\mathrm{s}^{-2}$ in velocity model SC1_II at the depth of 1.5 km reads (*Pšenčík et al., 2012, Eq. 19*)

$$\begin{array}{c} \\ 11 \\ 22 \\ 33 \\ 23 \\ 13 \\ 12 \end{array} \begin{array}{c} \begin{array}{cccccc} \quad 11 & \quad 22 & \quad 33 & \quad 23 & \quad 13 & \quad 12 \end{array} \\ \begin{pmatrix} 30.25 & 10.08 & 10.08 & 0.00 & 0.00 & 0.00 \\ & 30.25 & 10.08 & 0.00 & 0.00 & 0.00 \\ & & 30.25 & 0.00 & 0.00 & 0.00 \\ & & & 10.08 & 0.00 & 0.00 \\ & & & & 10.08 & 0.00 \\ & & & & & 10.08 \end{pmatrix} \end{array} . \tag{40}$$

The elements of the stiffness tensor (elastic moduli) are linear functions of depth. This means that, at depths between 0 km and 1.5 km, velocity model SC1_II is close to transversely isotropic, but is slightly tetragonal. For example, at the depth of 1.4 km, non–TI ratio (11) is

$$\rho_{(3)} = 0.000397 \ , \tag{41}$$

and the corresponding unit reference symmetry vector is

$$t_{i(3)} = (0.611611 \ \ 0.348810 \ \ 0.710115) \ . \tag{42}$$

We see that the medium is less transversely isotropic at the depth of 1.4 km than at the surface.

3.5. Orthorhombic medium

The orthorhombic medium by *Schoenberg and Helbig (1997)* was already used by *Bucha (2013c, 2014a)*. The density reduced stiffness tensor reads

$$\begin{array}{c} \\ 11 \\ 22 \\ 33 \\ 23 \\ 13 \\ 12 \end{array} \begin{array}{c} \begin{array}{cccccc} \quad 11 & \quad 22 & \quad 33 & \quad 23 & \quad 13 & \quad 12 \end{array} \\ \begin{pmatrix} 9.0000 & 3.6000 & 2.2500 & 0.0000 & 0.0000 & 0.0000 \\ & 9.8400 & 2.4000 & 0.0000 & 0.0000 & 0.0000 \\ & & 5.9375 & 0.0000 & 0.0000 & 0.0000 \\ & & & 2.0000 & 0.0000 & 0.0000 \\ & & & & 1.6000 & 0.0000 \\ & & & & & 2.1820 \end{pmatrix} \end{array} . \tag{43}$$

Non–TI ratio (11) determined for this medium is

$$\rho_{(3)} = 0.254022 \ , \tag{44}$$

and the corresponding unit reference symmetry vector is

$$t_{i(3)} = (0.000000 \ \ 0.000000 \ \ 1.000000) \ . \tag{45}$$

Ratios (12) read

$$\rho_{(2)} = 0.433103 \ , \qquad \rho_{(1)} = 0.436127 \ . \tag{46}$$

We see that this orthorhombic medium contains a considerable transversely isotropic component with the vertical axis of symmetry.

3.6. Triclinic medium

The triclinic medium by *Mensch and Rasolofosaon (1997)* was already used by *Bucha (2012, 2013a,b,c, 2014b, 2015, 2016)*. The density reduced stiffness tensor reads

$$\begin{array}{c} \\ 11 \\ 22 \\ 33 \\ 23 \\ 13 \\ 12 \end{array} \begin{array}{c} \begin{array}{cccccc} 11 & 22 & 33 & 23 & 13 & 12 \end{array} \\ \begin{pmatrix} 10.3 & 0.9 & 1.3 & 1.4 & 1.1 & 0.8 \\ & 10.6 & 2.1 & 0.2 & -0.2 & -0.6 \\ & & 14.1 & 0.0 & -0.5 & -1.0 \\ & & & 5.1 & 0.0 & 0.2 \\ & & & & 6.0 & 0.0 \\ & & & & & 4.9 \end{pmatrix} \end{array} . \tag{47}$$

Non–TI ratio (11) determined for this medium is

$$\rho_{(3)} = 0.275710 , \tag{48}$$

and the corresponding unit reference symmetry vector is

$$t_{i(3)} = (0.576529 \;\; 0.466767 \;\; 0.670629) . \tag{49}$$

Ratios (12) read

$$\rho_{(2)} = 0.353717 , \qquad \rho_{(1)} = 0.545104 . \tag{50}$$

We see that this triclinic medium contains a considerable transversely isotropic component with a tilted axis of symmetry.

4. APPLICATIONS

The possibility of determining whether a given stiffness tensor corresponds to a transversely isotropic medium may be very useful in selecting the method for calculating the wave field. If the medium is transversely isotropic or approximately transversely isotropic, we may use its symmetry vector or reference symmetry vector in tracing the SH and SV rays or the SH and SV reference rays (*Klimeš and Bulant, 2015*). The non–TI ratio, which identifies how much the given medium is transversely isotropic, and the unit reference symmetry vector can be determined according to Sections 2.1 and 2.2 of this paper.

If the reference symmetry vector is spatially varying, we also need its first–order spatial derivatives for ray tracing, and its second–order spatial derivatives for solving the corresponding equations of geodesic deviation (dynamic ray tracing equations). The first–order spatial derivatives of the reference symmetry vector can be determined according to Section 2.3 of this paper. The second–order spatial derivatives of the reference symmetry vector can be determined according to Section 2.4 of this paper.

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